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CONTACT PROBLEMS OF THE MECHANICS OF BODIES WITH ACCRETION*

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Contact problems of the mechanics of bodies with accretion are studied. A general formulation of the mixed problem is given for a viscoelastic ageing body during its continuous piecewise accretion. Complete systems of equations of the mixed problem are given in time intervals from the onset of loading to the onset of accretion, from the onset of accretion to the end of accretion, and beyond it.

The characteristic feature of the basic relations in the case of a body with continuous accretion is the use not of the usual equations of compatibility of the deformations and the Cauchy relations, but of their analogues in the rates of change of the corresponding quantities /1-3/. Moreover, the given previous histories of the deformation tensor of the accruing elements form, at the instant of attachment, specific initial and boundary conditions /2/ on the accruing surface. In particular, the total stress tensor associated with external loads and characterizing the tightness of attachment of the accruing elements is determined at the accruing surface /2, 3/. The instant of attachment of the new elements to the main body represents an important characteristic of the process. The set of instants of attachment completely determines the configuration of the accruing body at any instant of time. Equations of state of the theory of creep of the inhomogeneously ageing bodies are used /4, 5/. The equations reflect the fundamental specific features of the accretion process where the times of preparation and onset of loading play an important part.

A method of solving the mixed and initial-boundary value problems is given. Contact problems for a wedge under various methods of accretion are considered. Integral equations are derived and their solutions constructed. Numerical solutions of the contact problems for a wedge with accretion are given for the case when the influx of matter from outside results in increasing the wedge angle, and for an accruing quarter-plane. Qualitative and quantitative effects are discussed, especially the influence of the method and rate of accretion on the contact characteristics.

1. Formulation and solution of the mixed problem for an ageing, viscoelastic body with accretion. Let a homogeneous, viscoelastic ageing body manufactured at the instant $t=0$. occupy the region Ω_0 with surface S_0, and be stress-free up to the instant

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τ_0 of loading. From the instant of loading onwards, four types of boundary conditions are specified, in the general case, on the surface of the body: on $S_1(t)$ we specify the surface forces, and a stationary segment $S^* \subset S_1(t)$ of the surface exists on which the surface forces are equal to zero, on $S_2(t)$ we specify the displacements, on $S_3(t)$ the normal displacements and tangential forces, and on $S_4(t)$ the normal forces and tangential displacements. The segments of the surface on which different boundary conditions are specified do not intersect each other and together they occupy the whole surface of the body. The dependence of S_i on time t takes into account the possibility of motion of the lines on which the boundary conditions change, over the surface S_0 (motion of loads, stamps, etc.). If the surface of the body is not closed, then the behaviour of the stresses or displacements at infinity is specified.

At the instant $\tau_1 \geqslant \tau_0$ continuous accretion begins of the body with the elements made at the same time as the body. During the process of growth the body occupies the region $\Omega(t)$ with surface $S(t)$. The surface of accretion $S^*(t)$ ($S^*(\tau_1) = S^*$) moves in space, and during this process the segments $S_i(t)$ ($i = 1, \ldots, 4$) on which the boundary conditions are specified may change as the stationary surface is loaded along the freshly formed part of the body /2/. We shall assume that the total stress tensor specified at the surface of accretion is coordinated with the zero surface forces on $S^*(t)$, and the instant of application of the load to the accruing elements coincides with the instant of their application to the main body.

The body ceases to grow at the instant $\tau_2 > \tau_1$, and from this time onwards four types of boundary conditions are specified at the surface $S_1 = S(\tau_2)$ of the body occupying the region $\Omega_1 = \Omega(\tau_2)$ on the segments $S_i(t)$. A segment with specified zero surface forces need not exist, and the surface $S_1^* = S^*(\tau_2)$ may be loaded.

In what follows, we shall consider slow processes such that the inertial terms in the equations of equilibrium can be neglected. We assume that the mass forces are equal to zero.

We shall study the stress-deformation state of a homogeneous ageing viscoelastic body over the time interval $t \in [\tau_0, \tau_1]$. We have the following boundary value problem:

$$\begin{aligned}
&\nabla\cdot\sigma = 0 \qquad (1.1)\\
&\mathbf{x} \in S_1(t):\quad \mathbf{n}\cdot\sigma = \mathbf{p}_0, \quad \mathbf{x} \in S^* \subset S_1(t):\quad \mathbf{n}\cdot\sigma = 0\\
&\mathbf{x} \in S_2(t):\ \mathbf{u} = \mathbf{u}_0\\
&\mathbf{x} \in S_3(t):\ \mathbf{nn}\cdot\mathbf{u} = \mathbf{u}_1, \quad \mathbf{n}\cdot\sigma - \mathbf{n}\cdot\sigma\cdot\mathbf{nn} = \mathbf{p}_1\\
&\mathbf{x} \in S_4(t):\quad \mathbf{n}\cdot\sigma\cdot\mathbf{nn} = \mathbf{p}_2, \quad \mathbf{u} - \mathbf{nn}\cdot\mathbf{u} = \mathbf{u}_2\\
&\varepsilon = {}^1/_2\,[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]\\
&\sigma = E\,(1+\nu)^{-1}\,(\mathbf{I} + \mathbf{N}(\tau_0, t))\,[\varepsilon + \nu\,(1-2\nu)^{-1} I_1(\varepsilon)\,\mathbf{E}]\\
&(\mathbf{I} - \mathbf{L}(\tau_0, t)) = (\mathbf{I} + \mathbf{N}(\tau_0, t))^{-1}\\
&\mathbf{L}(\tau_0, t) f(t) = \int_{\tau_0}^{t} f(\tau)\,K(t,\tau)\,d\tau\\
&K(t,\tau) = E(\tau)\,(\partial/\partial\tau)\,[E^{-1}(\tau) + C(t,\tau)]
\end{aligned}$$

Here $\sigma, \varepsilon, \mathbf{u}, \mathbf{p}_i$ and $\mathbf{u}_i$ are the stress and strain tensors, the displacement vector, specified vectors of surface forces and displacements (the arguments $\mathbf{x}$ (the radius vector of a point of the body) and t (time) are omitted), $\mathbf{n}$ is the unit vector normal to the surface of the body, $E = E(t)$ is the modulus of instantaneous tensile strain, $I_1(\varepsilon)$ is the first invariant of the strain tensor, $\mathbf{E}$ is a unit tensor, $C(t,\tau)$ is the measure of tensile creep, $\mathbf{I}$ is an identity operator, ∇ is the del operator and Poisson's ratios of the instantaneous and creep strain are identical and are equal to ν.

Since the del operator and the operator $(\mathbf{I} - \mathbf{L}(\tau_0, t))$ are commutative, we can assume that $\sigma^\circ = (\mathbf{I} - \mathbf{L}(\tau_0, t))\,\sigma E^{-1}$, and transform the boundary value problem (1.1) to the following form more suitable for studying and constructing the solution:

$$\begin{aligned}
&\nabla\cdot\sigma^\circ = 0 \qquad (1.2)\\
&\mathbf{x} \in S_1(t):\ \mathbf{n}\cdot\sigma^\circ = (\mathbf{I} - \mathbf{L}(\tau_0, t))\,\mathbf{p}_0 E^{-1} = \mathbf{p}^\circ\\
&\mathbf{x} \in S^* \subset S_1(t):\ \mathbf{n}\cdot\sigma^\circ = 0; \quad \mathbf{x} \in S_2(t):\ \mathbf{u} = \mathbf{u}_0\\
&\mathbf{x} \in S_3(t):\quad \mathbf{nn}\cdot\mathbf{u} = \mathbf{u}_1, \quad \mathbf{n}\cdot\sigma^\circ - \mathbf{n}\cdot\sigma^\circ\cdot\mathbf{nn} = (\mathbf{I} - \mathbf{L}(\tau_0, t))\cdot\\
&\quad \mathbf{p}_1 E^{-1} = \mathbf{p}_1^\circ\\
&\mathbf{x} \in S_4(t):\quad \mathbf{n}\cdot\sigma\cdot\mathbf{nn} = (\mathbf{I} - \mathbf{L}(\tau_0, t))\,\mathbf{p}_2 E^{-1} = \mathbf{p}_2^\circ, \quad \mathbf{u} - \mathbf{nn}\cdot\\
&\quad \mathbf{u} = \mathbf{u}_2\\
&\varepsilon = {}^1/_2\,[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]\\
&\sigma^\circ = (1+\nu)^{-1}\,[\varepsilon + \nu\,(1-2\nu)^{-1} I_1(\varepsilon)\,\mathbf{E}]
\end{aligned}$$

Solving (1.2), we return to the true stresses by means of the formula ($R(t,\tau)$ is a resolvent of the kernel $K(t,\tau)$)

$$\sigma(\mathbf{x}, t) = E(t)\left[\sigma^\circ(\mathbf{x}, t) + \int_{\tau_0}^{t} \sigma^\circ(\mathbf{x}, \tau) R(t, \tau)\, d\tau\right] \tag{1.3}$$

Let us now consider directly the process of accretion of the main body $(\tau_1 \leqslant t \leqslant \tau_2)$. The mixed initial-boundary value problem for a growing body has the form

$$\begin{aligned}
&\nabla \cdot \sigma = 0 \\
&\mathbf{x} \in S_1(t)\colon \mathbf{n}\cdot\sigma = \mathbf{p}_0; \quad \mathbf{x} \in S_2(t)\colon \mathbf{u} = \mathbf{u}_0 \\
&\mathbf{x} \in S_3(t)\colon \quad \mathbf{nn}\cdot\mathbf{u} = \mathbf{u}_1, \quad \mathbf{n}\cdot\sigma - \mathbf{n}\cdot\sigma\cdot\mathbf{nn} = \mathbf{p}_1 \\
&\mathbf{x} \in S_4(t)\colon \quad \mathbf{n}\cdot\sigma\cdot\mathbf{nn} = \mathbf{p}_2, \quad \mathbf{u} - \mathbf{nn}\cdot\mathbf{u} = \mathbf{u}_2 \\
&\mathbf{x} \in S^*(t)\colon \sigma = \sigma^*, \quad t = \tau^*(\mathbf{x}) \\
&\varepsilon^{\cdot} = {}^1/_2\,[\nabla\mathbf{u}^{\cdot} + (\nabla\mathbf{u}^{\cdot})^T] \\
&\sigma = E(1+\nu)^{-1}(\mathbf{I} + \mathbf{N}(\tau_0(\mathbf{x}), t))\,[\varepsilon + \nu(1-2\nu)^{-1} I_1(\varepsilon)\,\mathbf{E}] \\
&\tau_0(\mathbf{x}) = \begin{cases} \tau_0, & \mathbf{x} \in \Omega_0 \\ \tau^*(\mathbf{x}), & \mathbf{x} \in \Omega(t) \setminus \Omega_0 \end{cases}
\end{aligned} \tag{1.4}$$

Here a dot denotes differentiation with respect to time t, $\tau^*(\mathbf{x})$ is the instant of attachment to the body of the element characterized by the radius vector $\mathbf{x}$, the operator $(\mathbf{I} - \mathbf{L}(\tau_0(\mathbf{x}), t))$ and its inverse $(\mathbf{I} + \mathbf{N}(\tau_0(\mathbf{x}), t))$ are obtained from (1.1) by replacing τ_0 by $\tau_0(\mathbf{x})$, $\sigma^* = \sigma^*(\mathbf{x})$ is the total stress tensor defined on $S^*(t)$ and satisfying the condition

$$\mathbf{x} \in S^*(t)\colon \mathbf{n}\cdot\sigma^* = 0 \quad (\tau_1 \leqslant t \leqslant \tau_2) \tag{1.5}$$

Relations (1.4) show that the process of accretion of the main body with new elements leads, generally, to the defining relations containing discontinuities at the boundary between the main body and its accrued parts. Transforming the initial-boundary value problem directly to the boundary value problem in terms of the rates of change of the corresponding quantities given in /2, 3/ leads, in this case, to considerable mathematical difficulties.

Let us investigate this approach. We shall write the equation of the surface of accretion $S^*(t)$ at $t = \tau_1$ $(S^*(\tau_1) = S^*)$ in the form $P = P(\mathbf{x}) = 0$ $(\mathbf{x} \in S^*)$ where $P < 0$ when $\mathbf{x} \in \Omega_0$ and $P \geqslant 0$ when $\mathbf{x} \in \Omega(t) \setminus \Omega_0$. We shall assume that P is a fairly continuous function such that $\nabla P \neq 0$ when $P = 0$ (i.e. there are no singularities at the surface S^*). We introduce the characteristic function $\theta(P)$ equal to unity when $P \geqslant 0$ and to zero when $P < 0$ /6/. We can now write the operator $\mathbf{T} = (\mathbf{I} - \mathbf{L}(\tau_0(\mathbf{x}), t))$ in the form

$$\begin{aligned}
&\mathbf{T} f(t) = (\mathbf{I} - \mathbf{L}(\tau^\circ(\mathbf{x}), t))\, f(t) - [1 - \theta(P)]\, \mathbf{L}^t(\tau_0, \tau_1)\, f(t) \\
&\mathbf{L}^t(\tau_0, \tau_1) f(t) = \int_{\tau_0}^{\tau_1} f(\tau) K(t, \tau)\, d\tau \\
&\tau^\circ(\mathbf{x}) = \tau_1 + \theta(P)\,[\tau^*(\mathbf{x}) - \tau_1], \quad \mathbf{x} \in S^*\colon \quad \tau^*(\mathbf{x}) = \tau_1
\end{aligned}$$

It can be shown that $\mathbf{T}\nabla\cdot\sigma E^{-1} = \nabla\cdot\mathbf{T}\sigma E^{-1}$, if condition (1.5) holds and

$$\mathbf{x} \in S^*\colon \quad \mathbf{n}\cdot\sigma = 0 \quad (\tau_0 \leqslant t \leqslant \tau_1) \tag{1.6}$$

Let us write $\sigma^\circ = \mathbf{T}\sigma E^{-1}$. Then $\delta(P)$ is a generalized function on a smooth surface and represents the analogue of the Dirac function /6, 7/, $\nabla\tau^\circ(\mathbf{x}) = \theta(P)\nabla\tau^*(\mathbf{x})$

$$\begin{aligned}
\nabla\cdot\sigma^\circ = \mathbf{T}\nabla\cdot\sigma E^{-1} &+ \theta(P)\nabla\tau^*(\mathbf{x})\cdot\sigma^*(\mathbf{x}) E^{-1}(\tau^*(\mathbf{x})) K(t, \tau^*(\mathbf{x})) + \\
&\delta(P)\nabla P\cdot\mathbf{L}^t(\tau_0, \tau_1)\sigma E^{-1}
\end{aligned}$$

The second term of this relation is equal to zero: by virtue of the definition of $\theta(P)$ when $\mathbf{x} \in \Omega_0$, and by virtue of (1.5) when $\mathbf{x} \in \Omega(t) \setminus \Omega_0$ (we recall that the direction of $\nabla\tau^*(\mathbf{x})$ is the same as the normal to the accretion surface $S^*(t)$ /2, 3/). The third term is equal to zero by virtue of (1.6), since the direction of ∇P is the same when $\mathbf{x} \in S^*$, as the normal to S^*. Conditions (1.5) and (1.6) represent the sufficient conditions for the commutativity of the operators $\mathbf{T}$ and the divergence operator on the set of tensor-valued functions defined in the region $\Omega(t)$, and have the following mechanical meaning: the surface of the initial solid towards which the mass flux is directed, is not placed under load until the process of accretion begins, nor is the accruing surface under load during its growth.

Let the operator $\mathbf{T}$ act on the expressions from (1.4) containing σ, having previously divided them by E. Then

$$\begin{aligned}
&\nabla\cdot\sigma^\circ = 0 \\
&\mathbf{x} \in S_1(t)\colon \quad \mathbf{n}\cdot\sigma^\circ = \mathbf{T}\mathbf{p}_0 = \mathbf{p}^\circ; \quad \mathbf{x} \in S_2(t)\colon \mathbf{u} = \mathbf{u}_0 \\
&\mathbf{x} \in S_3(t)\colon \quad \mathbf{nn}\cdot\mathbf{u} = \mathbf{u}_1, \ \mathbf{n}\cdot\sigma^\circ - \mathbf{n}\cdot\sigma^\circ\cdot\mathbf{nn} = \mathbf{T}\mathbf{p}_1 = \mathbf{p}_1^\circ
\end{aligned} \tag{1.7}$$

$$\begin{aligned}
&\mathbf{x} \in S_4(t):\ \mathbf{n}\cdot\boldsymbol{\sigma}^\circ\cdot\mathbf{n} = \mathbf{T}\mathbf{p}_2 = \mathbf{p}_2^\circ,\ \ \mathbf{u} - \mathbf{n}\mathbf{n}\cdot\mathbf{u} = \mathbf{u}_2\\
&\mathbf{x} \in S^*(t):\ \boldsymbol{\sigma}^\circ = \boldsymbol{\sigma}^{\circ*} = \boldsymbol{\sigma}^* E^{-1},\ \ \mathbf{n}\cdot\boldsymbol{\sigma}^* = \mathbf{0},\ \ t = \tau^*(\mathbf{x})\\
&\boldsymbol{\varepsilon}^{\cdot} = {}^1/_2\,[\nabla\mathbf{u}^{\cdot} + (\nabla\mathbf{u}^{\cdot})^T]\\
&\boldsymbol{\sigma}^\circ = (1+\nu)^{-1}\,[\boldsymbol{\varepsilon} + \nu\,(1-2\nu)^{-1}\,I_1(\boldsymbol{\varepsilon})\,\mathbf{E}]
\end{aligned}$$

Relations (1.7) represent an initial-boundary value problem with the operator stresses $\boldsymbol{\sigma}^\circ$, whose defining relations contain no discontinuities.

Let us transform the initial-boundary value problem (1.7) to the boundary value problem for the rates of $\boldsymbol{\sigma}^\circ, \boldsymbol{\varepsilon}$ and $\mathbf{u}$. To do this we differentiate with respect to t the equilibrium equations, the conditions on $S_i(t)$ $(i = 1, \ldots, 4)$, and equation of state. We obtain the boundary condition $S^*(t)$ by applying the divergence operator to initial-boundary condition $\boldsymbol{\sigma}^\circ(\mathbf{x}, \tau^*(\mathbf{x})) = \boldsymbol{\sigma}^{\circ*}(\mathbf{x})$ (see also /2/, in which case $\mathbf{n}\cdot\boldsymbol{\sigma}^{\circ\cdot} = \nabla\cdot\boldsymbol{\sigma}^{\circ*}\,|\nabla\tau^*(\mathbf{x})|^{-1}\ (\mathbf{x} \in S^*(t))$. We note that

$$\begin{aligned}
\nabla\cdot\boldsymbol{\sigma}^{\circ*} &= \nabla\cdot(\boldsymbol{\sigma}^*(\mathbf{x})\,E^{-1}(\tau^*(\mathbf{x}))) = \nabla\cdot\boldsymbol{\sigma}^*(\mathbf{x})\,E^{-1}(\tau^*(\mathbf{x})) -\\
&\nabla\tau^*(\mathbf{x})\cdot\boldsymbol{\sigma}^*(\mathbf{x})\,E^{-2}(\tau^*(\mathbf{x}))\,E^{\cdot}(\tau^*(\mathbf{x}))
\end{aligned}$$

where the second term is equal to zero by virtue of condition (1.5). Finally we obtain

$$\begin{aligned}
&\nabla\cdot\boldsymbol{\sigma}^{\circ\cdot} = \mathbf{0} &&(1.8)\\
&\mathbf{x} \in S_1(t):\ \mathbf{n}\cdot\boldsymbol{\sigma}^{\circ\cdot} = \mathbf{p}^{\circ\cdot} = \mathbf{G}\mathbf{p}_0,\ \ \mathbf{x} \in S_2(t):\ \mathbf{u}^{\cdot} = \mathbf{u}_0^{\cdot}\\
&\mathbf{x} \in S_3(t):\ \ \mathbf{n}\mathbf{n}\cdot\mathbf{u}^{\cdot} = \mathbf{u}_1^{\cdot},\ \ \mathbf{n}\cdot\boldsymbol{\sigma}^{\circ\cdot} - \mathbf{n}\cdot\boldsymbol{\sigma}^{\circ\cdot}\cdot\mathbf{n}\mathbf{n} = \mathbf{p}_1^{\circ\cdot} = \mathbf{G}\mathbf{p}_1\\
&\mathbf{x} \in S_4(t):\ \ \mathbf{n}\cdot\boldsymbol{\sigma}^{\circ\cdot}\cdot\mathbf{n}\mathbf{n} = \mathbf{p}_2^{\circ\cdot} = \mathbf{G}\mathbf{p}_2,\ \ \mathbf{u}^{\cdot} - \mathbf{n}\mathbf{n}\cdot\mathbf{u}^{\cdot} = \mathbf{u}_2^{\cdot}\\
&\mathbf{x} \in S^*(t):\ \mathbf{n}\cdot\boldsymbol{\sigma}^{\circ\cdot} = \nabla\cdot\boldsymbol{\sigma}^* E^{-1}\,|\nabla\tau^*|^{-1},\ \ t = \tau^*(\mathbf{x})\\
&\boldsymbol{\varepsilon}^{\cdot} = {}^1/_2\,[\nabla\mathbf{u}^{\cdot} + (\nabla\mathbf{u}^{\cdot})^T]\\
&\boldsymbol{\sigma}^{\circ\cdot} = (1+\nu)^{-1}\,[\boldsymbol{\varepsilon}^{\cdot} + \nu\,(1-2\nu)^{-1}\,I_1(\boldsymbol{\varepsilon}^{\cdot})\,\mathbf{E}]\\
&\mathbf{G}f(t) = \frac{f^{\cdot}(t)}{E(t)} + \int_{\tau_0(\mathbf{x})}^{t} \frac{\partial f(\tau)}{\partial\tau}\frac{\partial C(t,\tau)}{\partial t}\,d\tau + f(\tau_0(\mathbf{x}))\,\frac{\partial C(t,\tau_0(\mathbf{x}))}{\partial t}
\end{aligned}$$

Thus the initial-boundary value problem of the accretion of a viscoelastic solid with ageing (1.4) reduces to the boundary value problem (1.8) for the rates of displacement $\mathbf{u}^{\cdot}$, strain $\boldsymbol{\varepsilon}^{\cdot}$ and operator stresses $\boldsymbol{\sigma}^{\circ\cdot}$ where the time t is a parameter. We note that the boundary value problems (1.2) and (1.8) are mathematically equivalent, since the boundary conditions on $S_1(t)$ and $S^*(t)$ in problem (1.8) are identical, i.e., just as in the problem with a fixed boundary only four types of boundary conditions are specified on the surface.

After solving (1.8) we can find the stresses and displacements in a growing solid, for $\tau_1 \leqslant t \leqslant \tau_2$, from the formulas

$$\begin{aligned}
&\mathbf{x} \in \Omega_0:\ \ \boldsymbol{\sigma}(\mathbf{x}, t) = E(t)\Big\{\Big[\frac{\boldsymbol{\sigma}(\mathbf{x},\tau_1)}{E(\tau_1)} - &&(1.9)\\
&\int_{\tau_0}^{\tau_1}\frac{\boldsymbol{\sigma}(\mathbf{x},\tau)}{E(\tau)}K(\tau_1,\tau)\,d\tau\Big]\Big[1 + \int_{\tau_1}^{t}R(t,\tau)\,d\tau\Big] +\\
&\int_{\tau_0}^{\tau_1}\frac{\boldsymbol{\sigma}(\mathbf{x},\tau)}{E(\tau)}K(t,\tau)\,d\tau + \int_{\tau_1}^{t}\boldsymbol{\sigma}^{\circ\cdot}(\mathbf{x},\tau)\,d\tau +\\
&\int_{\tau_1}^{t}\Big[\int_{\tau_0}^{\tau_1}\frac{\boldsymbol{\sigma}(\mathbf{x},z)}{E(z)}K(\tau,z)\,dz + \int_{\tau_1}^{\tau}\boldsymbol{\sigma}^{\circ\cdot}(\mathbf{x},z)\,dz\Big]R(t,\tau)\,d\tau\Big\}\\
&\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x},\tau_1) + \int_{\tau_1}^{t}\mathbf{u}^{\cdot}(\mathbf{x},\tau)\,d\tau\\
&\mathbf{x} \in \Omega(t)\setminus\Omega_0:\ \boldsymbol{\sigma}(\mathbf{x}, t) = E(t)\Big\{\frac{\boldsymbol{\sigma}^*(\mathbf{x})}{E(\tau^*(\mathbf{x}))}\Big[1 + \int_{\tau^*(\mathbf{x})}^{t}R(t,\tau)\,d\tau\Big] +\\
&\int_{\tau^*(\mathbf{x})}^{t}\Big[\boldsymbol{\sigma}^{\circ\cdot}(\mathbf{x},\tau) + \int_{\tau^*(\mathbf{x})}^{\tau}\boldsymbol{\sigma}^{\circ\cdot}(\mathbf{x},z)\,dz\,R(t,\tau)\Big]d\tau\Big\}\\
&\mathbf{u}(\mathbf{x}, t) = \int_{\tau^*(\mathbf{x})}^{t}\mathbf{u}^{\cdot}(\mathbf{x},\tau)\,d\tau
\end{aligned}$$

In deriving (1.9) we have used relations of the form

$$\omega(\mathbf{x}, t) = \omega(\mathbf{x}, \tau^\circ(\mathbf{x})) + \int_{\tau^\circ(\mathbf{x})}^{t}\omega^{\cdot}(\mathbf{x},\tau)\,d\tau$$

inverses of the Volterra operators, and the known information concerning the stresses and displacements at $\tau_0 \leqslant t \leqslant \tau_1$ obtained in the previous stage. The initial values of the displacements in the growing part of the body $\mathbf{u}(\mathbf{x}, \tau^*(\mathbf{x}))$ were assumed to be equal to zero /2, 3/.

Relations (1.8) and (1.9) show that the stress-strain state of a growing viscoelastic body is affected by the whole history of its loading and growth. Moreover, these relations can be used to find the loading modes for which the process of accretion will not significantly affect the state of initial body, and the accrued part will be practically undeformed. Indeed, if we assume that only the surface of initial body is loaded, the forces are stationary, the growth process does not produce any tension and the time at which accretion begins is much later than the onset of loading, then using the property of limited creep of a viscoelastic material $\lim_{t\to\infty} \partial C(t, \tau)/\partial t = 0$ we arrive, using (1.8) and (1.9), at the conclusion formulated above. We reach the same conclusions by analysing the mode of loading in which the forces remain constant for a long time before the onset of growth, irrespective of their previous changes (we assume that having reached steady-state values, the forces no longer change).

The effects discussed here have a clear mechanical meaning. Under the conditions of limited creep a viscoelastic body acted upon by stationary forces will practically cease to deform. The subsequent accretion of the unstressed elements leads to a situation in which the interaction between the initial body and the accrued parts becomes insignificant.

Let us now suppose that the growth of the body stops at the instant τ_2. The body occupies at this instant the region Ω_1 with surface S_1 on which four types of boundary conditions are specified, and $S^*(\tau_2) = S_1^* \subset S_i(t)$ $(i = 1, \ldots, 4)$. In this case the boundary value problem has the form (1.4) where we have no initial-boundary condition on $S^*(t)$ and $\tau^*(\mathbf{x}) = \tau_2$ when $\mathbf{x} \in S_1^*$ $(t \geqslant \tau_2)$. After the transformations analogous to those carried out above for the initial boundary value problem of accretion, the boundary value for determining the stress-deformation state after the termination of the growth, takes the form (1.8) in which we omit the condition at the growing surface. The stresses and displacements are found in this case from the formulas

$$\boldsymbol{\sigma}(\mathbf{x}, t) = E(t)\Big\{\Big[\frac{\boldsymbol{\sigma}(\mathbf{x}, \tau_2)}{E(\tau_2)} - \int_{\tau_0(\mathbf{x})}^{\tau_2} \frac{\boldsymbol{\sigma}(\mathbf{x}, \tau)}{E(\tau)} K(\tau_2, \tau)\, d\tau\Big] \times \tag{1.10}$$

$$\Big[1 + \int_{\tau_2}^{t} R(t, \tau)\, d\tau\Big] + \int_{\tau_0(\mathbf{x})}^{\tau_2} \frac{\boldsymbol{\sigma}(\mathbf{x}, \tau)}{E(\tau)} K(t, \tau)\, d\tau +$$

$$\int_{\tau_2}^{t} \boldsymbol{\sigma}^{\circ\cdot}(\mathbf{x}, \tau)\, d\tau + \int_{\tau_2}^{t} \Big[\int_{\tau_0(\mathbf{x})}^{\tau_2} \frac{\boldsymbol{\sigma}(\mathbf{x}, z)}{E(z)} K(\tau, z)\, dz +$$

$$\int_{\tau_2}^{\tau} \boldsymbol{\sigma}^{\circ\cdot}(\mathbf{x}, z)\, dz\Big] R(t, \tau)\, d\tau\Big\}$$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, \tau_2) + \int_{\tau_2}^{t} \mathbf{u}^{\cdot}(\mathbf{x}, \tau)\, d\tau$$

Note 1. Using the method proposed above, we can study the piecewise continuous process of growth of a viscoelastic body with ageing, with any number of starts and stops during the growth process. The areas of the surface of the body on which the matter accrues, should not become loaded. The growth surface must also be load-free during the accretion process. If the accretion continues on some segment of this surface after the process has stopped, this segment must also remain load-free. The problem with n instants of the onset of accretion (and therefore with n terminations), can be reduced to the study of $2n + 1$ problems of the same type when the time is a parameter and the stress-strain state of the viscoelastic body is re-established according to known formulas of the form (1.9) and (1.10).

2. On the interaction of a viscoelastic ageing wedge with a smooth rigid stamp with side accretion. Consider a homogeneous, ageing wedge with aperture angle of α_0, constructed at the zero instant (Fig.1). At the instant τ_0 a smooth rigid stamp with foundation described by the function $g(r)$, begins to imbed into one of the faces of the wedge over the segment $a \leqslant r \leqslant b$. A force $P(t)$ and moment $M(t)$ act on the stamp, with eccentricity equal to $e(t)$. The other face of the wedge is stress-free.

The load-free face of the wedge at the instant τ_1 begins to accrue non-stressed elements, so that the opening angle $\alpha(t)$ of the wedge changes with time. We shall call the accretion following this rule side accretion. At the instant τ_2 the wedge ceases to grow, its aperture angle up to this instant is equal to $\alpha_1 < 2\pi$, and the fact accruing the matter is stress-free even when $t \geqslant \tau_2$.

We shall assume that the accruing elements are fabricated at the same time as the starting body, and the displacements of the point of the wedge tend to zero as $r \to \infty$. The wege is under the conditions of plane deformation.

Let us replace the stamp by a normal distributed load $q(r, t)$ acting over the same segment. Using Sect.1, we arrive at the solution of three boundary value problems: one for σ° and $\mathbf{u}$ in the interval from the instant of loading to the beginning of accretion, and $\sigma^{\circ\cdot}$ and $\mathbf{u}^\cdot$ from the beginning of accretion to its cessation and from cessation to any, time as long as desired. Using the integral Mellin transform /8/ and contour integration /9/ we obtain, after some reduction, a relation connecting the displacement $u_\theta(r, \theta, t) \times (\tau_0 \leqslant t \leqslant \tau_1)$ with the operator value of the load $q^\circ(r, t)$, and the rate of displacement $u_\theta^\cdot(r, \theta, t)$ $(\tau_1 \leqslant t \leqslant \tau_2$ and $t \geqslant \tau_2)$ with the rate of operator loading $q^{\circ\cdot}(r, t)$. Equating $u_\theta(r, \theta, t)$ and $u_\theta^\cdot(r, \theta, t)$ at $\theta = 0$ and $a \leqslant r \leqslant b$ with the displacement and rate of displacement of the stamp regarded as a rigid unit, we obtain the integral equations of the problem in the form

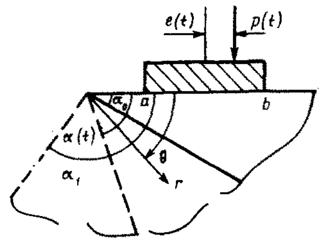

Fig.1

$$\int_a^b k(\rho, r, \alpha_0) q^\circ(\rho, t)\, d\rho = \mu[\omega(t) r - g(r)] \quad (\tau_0 \leqslant t \leqslant \tau_1) \tag{2.1}$$

$$\int_a^b k(\rho, r, \alpha(t)) q^{\circ\cdot}(\rho, t)\, d\rho = \mu\omega^\cdot(t) r \quad (\tau_1 \leqslant t \leqslant \tau_2) \tag{2.2}$$

$$\int_a^b k(\rho, r, \alpha_1) q^{\circ\cdot}(\rho, t)\, d\rho = \mu\omega^\cdot(t) r \quad (t \geqslant \tau_2) \tag{2.3}$$

$$k(\rho, r, \lambda) = \int_0^\infty \frac{p \sin 2\lambda + \operatorname{sh} 2\lambda p}{2p(\operatorname{sh}^2 \lambda p - p^2 \sin^2 \lambda)} \cos p\eta\, dp + \frac{1}{2}\eta\zeta(\lambda)$$

$$\zeta(\lambda) = {}^1/_2\pi\,(2\lambda + \sin 2\lambda)/(\lambda^2 - \sin^2\lambda)$$

$$\eta = \ln(\rho/r), \quad \mu^{-1} = 2(1 - \nu^2)/\pi$$

where $\omega(t)$ is the angle of rotation of the stamp.

We note that the integral in the expression for $k(\rho, r, \lambda)$ exists only in the sense of generalized functions, and its regularization is determined, apart from an arbitrary functional concentrated at zero /6/. This implies that the displacement and the rate of displacement of the points of the wedge are determined, apart from an arbitrary additive function of time, and for this reason the right-hand sides of (2.1)-(2.3) contain no terms characterizing the settling and the rate of settling of the stamp.

Eqs.(2.1)-(2.3) should be supplemented by conditions of equilibrium of the stamp, valid at any instant of time

$$P(t) = \int_a^b q(r, t)\, dr, \quad M(t) = \int_a^b \left(r - \frac{a+b}{2}\right) q(r, t)\, dr \tag{2.4}$$

Two situations are possible in the contact problem in question: 1) the force and angle of rotation are both known, and the contact pressure and eccentricity of the point of application of the force (moment) are to be determined: 2) the force and the eccentricity (moment) are known, and the contact stresses and angle of rotation are to be determined.

The first condition of (2.4) can be conveniently transformed, in both cases, to the form ($\mathbf{G}^\circ = \mathbf{G}$ when $\mathbf{x} \in \Omega_0$ (see (1.8))

$$\int_a^b q^\circ(\rho, t)\, d\rho = (\mathbf{I} - \mathbf{L}(\tau_0, t)) P(t) E^{-1} = P^\circ(t) \quad (\tau_0 \leqslant t \leqslant \tau_1) \tag{2.5}$$

$$\int_a^b q^{\circ\cdot}(\rho, t)\, d\rho = \mathbf{G}^\circ P(t) \quad (t \geqslant \tau_1)$$

The second condition of (2.4) has, in case 1), the form

$$\int_a^b \rho q(\rho, t)\, d\rho = M(t) + \frac{a+b}{2} P(t) \quad (t \geqslant \tau_0) \tag{2.6}$$

$$e(t) = M(t) P^{-1}(t)$$

and in case 2) it is taken in the form

$$\int_a^b \rho q^\circ(\rho, t)\, d\rho = (\mathbf{I} - \mathbf{L}(\tau_0, t)) \left(M(t) + \frac{a+b}{2} P(t)\right) E^{-1} \tag{2.7}$$

$$(\tau_0 \leqslant t \leqslant \tau_1)$$

$$\int_a^b \rho q^{\circ'}(\rho, t)\,d\rho = G^\circ\Big(M(t) + \frac{a+b}{2}P(t)\Big) \quad (t \geqslant \tau_1)$$

Let us investigate the equation which includes (2.1)-(2.3) as special cases, with an additional condition of the form (2.5). We shall write them in the form

$$\frac{2(1-\nu^2)}{\pi}\int_a^b k(\rho, r, \lambda)\,\varphi(\rho, t)\,d\rho = \psi(t)\,r - h(r) \tag{2.8}$$

$$\int_a^b \varphi(\rho, t)\,d\rho = \varkappa(t)$$

Let us approximate the factor accompanying the integrand of the kernel $k(\rho, r, \lambda)$ represented in the form of a quotient (see (2.3), etc.) by the function $\operatorname{cth}[\pi\zeta^{-1}(\lambda)p]\,p^{-1}$. The error of such an approximation does not exceed 15% for all $p \in [0, \infty]$ and $0 < \lambda \leqslant 2\pi$, and in the case of $\pi/2 \leqslant \lambda \leqslant 2\pi$ it does not exceed 5% /10, 11/. Then, using the regularization of the integral /12/

$$\int_0^\infty \frac{\operatorname{cth}\pi\zeta^{-1}(\lambda)\,p}{p}\cos p\eta\,dp = -\ln\Big[2\operatorname{sh}\frac{\zeta(\lambda)\,\eta}{2}\Big]$$

we obtain the expression for the kernel of Eq.(2.8)

$$k(\rho, r, \lambda) = -\ln\left[(\rho^{\zeta(\lambda)} - r^{\zeta(\lambda)})/\rho^{\zeta(\lambda)}\right] \tag{2.9}$$

Substituting (2.9) into (2.8) and differentiating the resulting expression with respect to r (we shall denote this operation by a prime), we obtain (see also /10/)

$$\frac{2(1-\nu^2)}{\pi}\int_a^b \frac{\zeta(\lambda)\,r^{\zeta(\lambda)-1}}{\rho^{\zeta(\lambda)} - r^{\zeta(\lambda)}}\,\varphi(\rho, t)\,d\rho = \psi(t) - h'(r) \tag{2.10}$$

Changing the variables we can reduce Eq.(2.10) to a well-known singular integral equation (see e.g. /13/).

Taking into account the additional condition (2.8), we obtain the solution of (2.10) in the form

$$\varphi(r, t) = \frac{\zeta(\lambda)\,r^{\zeta(\lambda)-1}}{\pi\left[(r^{\zeta(\lambda)} - a^{\zeta(\lambda)})(b^{\zeta(\lambda)} - r^{\zeta(\lambda)})\right]^{1/2}}\Big\{\varkappa(t) - \frac{1}{2(1-\nu^2)}\int_a^b \left[(\rho^{\zeta(\lambda)} - a^{\zeta(\lambda)})(b^{\zeta(\lambda)} - \rho^{\zeta(\lambda)})\right]^{1/2}\frac{\psi(t) - h'(\rho)}{\rho^{\zeta(\lambda)} - r^{\zeta(\lambda)}}\,d\rho\Big\} \tag{2.11}$$

In order to obtain the solution of (2.1), taking into account the additional condition (2.5) $(\tau_0 \leqslant t \leqslant \tau_1)$, we must put $\varphi(r, t) = q^\circ(r, t)$, $\lambda = \alpha_0$, $\varkappa(t) = P^\circ(t)$, $\psi(t) = \omega(t)$, $h'(\rho) = g'(\rho)$ in relation (2.11). If the angle of rotation of the stamp is given, then the operator stresses $q^\circ(r, t)$ will be known and it will only remain to re-establish the contact stresses with the help of formula (1.3) and to find the eccentricity of the point of application of the force using relations (2.6). If the moment is known, then the expression for $q^\circ(r, t)$, obtained from (2.11) and the first condition of (2.7) will together form a system of equations for determining $q^\circ(r, t)$ and $\omega(t)$, and after solving it we can use (1.3) to find the stresses $q(r, t)$.

We obtain the solution of (2.2) with condition (2.5) $(\tau_1 \leqslant t \leqslant \tau_2)$, by putting $\varphi(r, t) = q^{\circ*}(r, t)$, $\lambda = \alpha(t)$, $\varkappa(t) = G^\circ P(t)$, $\psi(t) = \omega^*(t)$, $h'(\rho) = 0$ in (2.11). In case 1) we find $q(r, t)$ and $e(t)$ using formulas (1.9) and (2.6), and in case 2) we find $q(r, t)$ and $\omega(t)$ using the second condition of (2.7) and (1.9).

When $(t \geqslant \tau_2)$, the solution of (2.3) and (2.5) can be formed by putting, unlike the case discussed above, $\lambda = \alpha_1$. We find $q(r, t)$ and $e(t)$ or $q(r, t)$ and $\omega(t)$, using formulas (1.10) instead of (1.9).

Note 2. In the case of numerical calculations, we must solve integrals of the type

$$I = \int_a^b \frac{\xi^\gamma}{[(\xi - a)(b - \xi)]^{1/2}}\,d\xi$$

The integrals can be reduced to definite integrals containing no singularities, and the following formula holds:

$$I = \frac{a^\gamma + b^\gamma}{2}\pi + \int_0^1 \left\{\left[\frac{a+b+(b-a)z}{2}\right]^\gamma + \left[\frac{a+b-(b-a)z}{2}\right]^\gamma - b^\gamma - a^\gamma\right\}(1-z^2)^{-1/2}dz$$

Example 1. Let us consider a contact problem for a wedge, with side accretion of non-stressed elements. The initial aperture angle is $\alpha_0 = \pi/2$, becoming $\alpha_1 = \pi$ at the instant of termination. The base of the stamp remains flat, the constant force and angle of rotation are specified, while the contact stresses and the eccentricity of the application of the force which ensure that the stamp is not misaligned, are determined. We assume that the growth rate is constant, in which case the instant of the onset τ_1 and termination τ_2 of growth completely determine the function $\alpha(t)$ and thus the configuration of the body at any instant of time. We choose concrete as the material of the wedge, and change to dimensionless quantities using the formulas

$$\begin{gathered} r^* = ra^{-1},\ \ \rho^* = \rho a^{-1},\ \ e^*(t^*) = e(t)a^{-1},\ \ t^* = t\tau_0^{-1} \\ \tau^* = \tau\tau_0^{-1},\ \tau_1^* = \tau_1\tau_0^{-1},\ \tau_2^* = \tau_2\tau_0^{-1},\ c = ba^{-1} \\ \alpha^*(t^*) = \alpha(t),\ q^*(r^*, t^*) = q(r, t)E^{-1}(t),\ P^*(t^*) = P(t)E^{-1}(t)\,a^{-1} \\ K^*(t^*, \tau^*) = K(t, \tau)\tau_0,\ g^*(r^*) = g(r)a^{-1},\ \omega^*(t^*) = \omega(t) \end{gathered} \tag{2.12}$$

(henceforth we shall omit the asterisks). Assuming that the elastic characteristics are constant, we shall specify the following values of the functions and parameters /5, 14, 15/ (we note that in the case in question the solution does not depend on Poisson's ratio):

$$C(t, \tau) = (C_0 + A_0 e^{-\beta\tau})(1 - e^{-\gamma(t-\tau)}),\ \ \alpha(t) = {}^1/_2\pi\,(t + \tau_2 - 2\tau_1)/(\tau_2 - \tau_1),\ \ P(t) = 1,\ \ C_0 = 0.5522,\ A_0 = 4,\ \beta = 0.31,\ \gamma = 0.6,\ c = 9,$$
$$\omega(t) = 0,\ \ g(r) = 0,\ \ \tau_0 = 1$$

Let us consider the case of a $(\tau_1 = 2, \tau_2 = 10, \alpha^{\cdot}(t) = \pi/16)$ and rapid $(\tau_1 = 2, \tau_2 = 4, \alpha^{\cdot}(t) = \pi/4)$ side accretion on the wedge. From now on the dot-dash lines in the figures will represent the basic characteristics during the slow growth, and dashed lines during the rapid growth (at a rate four times as high as that of the slow growth), and solid lines will cover the period from the onset of loading to the onset of accretion.

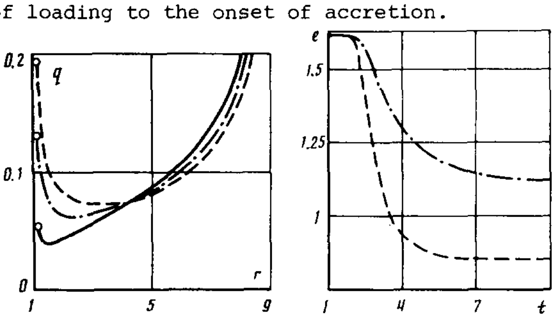

Fig.2

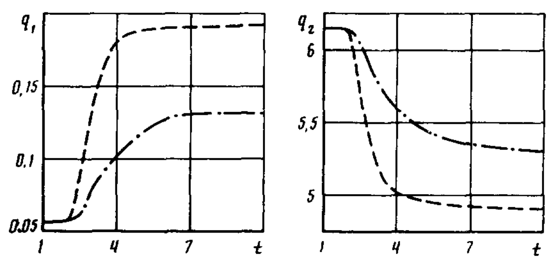

Fig.3

Fig.2 shows the limit distributions of contact pressures under the stamp q as $t \to \infty$ at different growth rates (dashed and dash-dot lines), and the distribution from the instant τ_0 to the instant τ_1, constant over this time interval (solid line). The distribution functions have singularities at the stamp edges, but in order to make the graphs easier to construct, we have eliminated the region $r < 0,1$ and indicated the values of the stresses at $r = 0,1$ by points. The figure also shows the dependence of the eccentricity e of the point of application of the force on time t.

Fig.3 shows the variation with time of the contact stresses q_1 at the left edge of the stamp $(r = 0,1)$ and q_2 at the right edge $(r = 8,9)$ for the two processes in question.

We see that the processes characterized by the rate of growth show substantial qualitative and quantitative differences between each other. During the rapid growth the most intense change in the stresses and the eccentricity occurs in the interval $t \in [\tau_1, \tau_2]$ and continues after the growth has ceased. The stresses, e.g. at the left edge, increase by a factor of 3.5 and the eccentricity decreases by almost a half.

When the growth is slow, the characteristics change more smoothly and, beginning at a certain instant of time $t^0 \approx 8$, the stress-strain state of the body becomes practically indifferent to the process of growth, i.e. under the conditions of limited creep the constant force acting on the stamp becomes exhausted and further growth or its termination does not alter the already established values of the stresses and the eccentricity. We note for comparison that in case of a slow growth the stresses at the left edge increase by a factor of 2.4, and the eccentricity decreases by a factor of 1.5. By studying the process of slow growth we also broaden our understanding of the law governing the formation of the displacement and stress fields in a growing viscoelastic body acted upon by stationary forces discussed in Sect.1.

Irrespective of the fact that the body has the form of a half-plane at the instant of termination, the distribution of contact pressures under the stamp is not symmetrical at any $t \geqslant \tau_2$, and the eccentricity of the point of application of the force is very far from zero, i.e. the idea of a body which has grown to a half-plane may lead directly to incommensurable values of the fundamental characteristics.

Note 3. Generally speaking, we can select, from the infinite manifold of parameters of the process of accretion, a case when the previous history of the strain tensor of the accruing elements ensures the compatibility of the strains over the whole body. The contact characteristics in the problem for a body grown up to a half-plane will be identical, provided the strains are compatible, with the characteristics of the contact problem for a half-plane. The realization of such a process in practice however, is questionable. Besides, this case is degenerate in the theory of accretion, since it leads to the equations and boundary conditions usually encountered in the mechanics of deformable solids, in a region varying with time.

3. The contact problem for a translationally accruing wedge. Let us now consider a viscoelastic aging wedge with aperture angle $0 < \alpha < \pi$, made at the zero instant (plane deformation). A smooth rigid stamp begins to impress itself into it at the instant τ_0 on the segment $a \leqslant x \leqslant b$ under the action of a force $P(t)$ with eccentricity of its application $e(t)$. The base of the stamp is described by the function $g(x)$ and the wedge surface is load-free with the exception of the segment $a \leqslant x \leqslant b$. At the instant τ_1 the stress-free elements manufactured at the same time as the wedge, begin to accrue on the surface. The free edge moves while remaining parallel to its initial position, and the general configuration of the body as a wedge with a fixed aperture angle, is preserved. We shall call such an accretion translational. The termination occurs at the instant τ_2. The process of accretion discussed here is determined by the function $f(t)$, which characterizes the distance along the x axis from the apex of the starting wedge to the apex of the accrued wedge. Clearly, $f(t) = 0$ when $\tau_0 \leqslant t \leqslant \tau_1$, and we shall also write $f(\tau_2) = f_1$ (Fig.4).

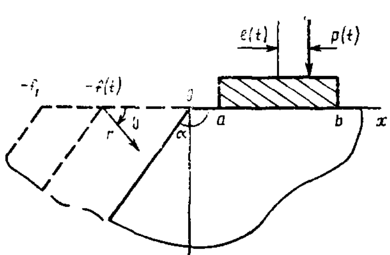

Fig.4

As before, we replace the stamp by a normal distributed load $q(x, t)$ and change to a moving (r, θ) coordinate system /16/. Following the procedure of Sect.2, we establish the relations connecting the displacements and rates of displacement with the operator loads and their rates. Returning to the initial fixed (x, y) coordinate system, we obtain integral equations of the problem in the form (in what follows, $q(x, t)$ are the contact pressure and $\omega(t)$ is the angle of rotation of the stamp)

$$\int_a^b k^\circ(\xi, x) q^\circ(\xi, t)\, d\xi = \mu[\omega(t) x - g(x)] \quad (\tau_0 \leqslant t \leqslant \tau_1) \tag{3.1}$$

$$\int_a^b k^\circ(\xi + f(t),\ x + f(t)) q^{\circ\cdot}(\xi, t)\, d\xi = \mu \omega^\cdot(t) x \quad (\tau_1 \leqslant t \leqslant \tau_2) \tag{3.2}$$

$$\int_a^b k^\circ(\xi + f_1, x + f_1)\, q^\circ(\xi, t)\, d\xi = \mu\omega^\cdot(t)\, x \quad (t \geqslant \tau_2) \tag{3.3}$$

with additional conditions of the form (2.4) where r should be replaced by x. Moreover, $k^\circ(\xi, x) = k(\xi, x, \alpha)$ (see (2.3) etc.).

Two situations are also possible in the contact problem with translational accretion, in which we specify the force, and either the angle of rotation or the eccentricity. The additional condition will now transform in accordance with (2.5)-(2.7) where ρ will be replaced by ξ (when referring in Sect.3 to relations (2.5)-(2.7), we will be assuming that such a substitution has been made). Subsequent investigation is carried out by means of a change of variable, an approximation of the type (2.9), and the solution of a known singular integral equation which, by virtue of the additional condition imposed on the forces, has the form (2.11).

Solving Eq.(3.1) with condition (2.5) $(\tau_0 \leqslant t \leqslant \tau_1)$ we find, putting in (2.11) $\varphi(r, t) = q^\circ(x, t)$, $\varkappa(t) = P^\circ(t)$, $\psi(t) = \omega(t)$, $\lambda = \alpha$, $h'(\rho) = g'(\xi)$ (the prime denotes differentiation with respect to ξ), $r = x$, $\rho = \xi$. When the angle of rotation is specified (case 1), relation (2.11) yields an expression for $q^\circ(x, t)$. After this we use formula (1.3) to obtain the distribution function of the contact pressures $q(x, t)$, and we substitute it into (2.6) to obtain the eccentricity of the application of the force. When the eccentricity is given (case 2), we must use the expression for $q^\circ(x, t)$, the first condition of (2.7) and formula (1.3), and here we determine $q(x, t)$ and the angle of rotation $\omega(t)$.

The solution of Eq.(3.2) with additional condition (2.5) $(\tau_1 \leqslant t \leqslant \tau_2)$, is obtained by putting in $\varphi(r, t) = q^{\circ\cdot}(x, t), \varkappa(t) = G^\circ P(t), \psi(t) = \omega^\cdot(t)$, $\lambda = \alpha$, $h'(\rho) = 0$, $r = x + f(t)$, $\rho = \xi + f(t)$ (2.11). In case 1 we determined $q(x, t)$ and $e(t)$ using (1.9) and (2.6), and in case 2 we seek $q(x, t)$ and $\omega(t)$ using (2.7) $(t \geqslant \tau_2)$ and (1.9).

The solution of (3.3) and (2.5) $(t \geqslant \tau_1)$ is obtained by putting in $r = x + f_1$, $\rho = \xi + f_1$ (2.11) (the remaining functions have the same form as in case of Eq.(3.2)). The contact stresses, the eccentricity or angle of rotation are found in the same manner as before, but we use (1.10) instead of (1.9).

Note 4. Using Sects.2 and 3, we find no fundamental difficulties when considering the contact problem for a wedge with complex accretion of the stressed elements. We merely note that any accretion during which the body retains its wedge form, can be described by a combination of the angular and translational method.

Note 5. When using numerical methods, it is sometimes more convenient to use, instead of formulas (1.9) and (1.10),

$$\sigma(\mathbf{x}, t) = E(t)\left\{\frac{\sigma(\mathbf{x}, \tau_0(\mathbf{x}))}{E(\tau_0(\mathbf{x}))}\left[1 + \int_{\tau_0(\mathbf{x})}^{t} R(t, \tau)\, d\tau\right] + \int_{\tau_0(\mathbf{x})}^{t} \sigma^{\circ\cdot}(\mathbf{x}, \tau)\, d\tau + \int_{\tau_0(\mathbf{x})}^{t}\int_{\tau_0(\mathbf{x})}^{\tau} \sigma^{\circ\cdot}(\mathbf{x}, z)\, dz R(t, \tau)\, d\tau\right\} \tag{3.4}$$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, \tau_0(\mathbf{x})) + \int_{\tau_0(\mathbf{x})}^{t} \mathbf{u}^\cdot(\mathbf{x}, \tau)\, d\tau$$

We must remember, however, that relations (3.4) must be transformed in accordance with the case in question (direct accretion, or after the termination), since the expressions for the rates of operator stresses and displacements at various stages of the process are obtained from the solutions of different problems.

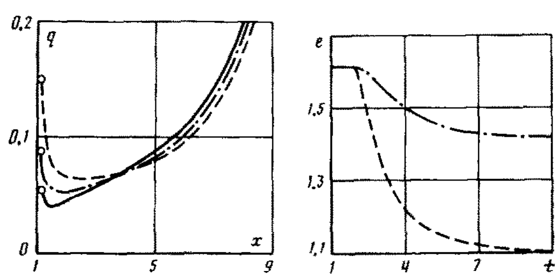

Fig.5

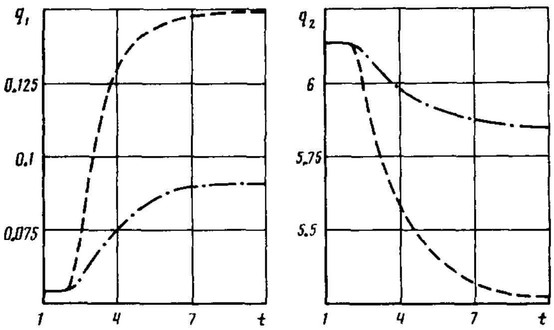

Fig.6

Example 2. The contact problem for an accruing quarter-plane. We shall consider the problem of translational growth on a quarter-plane of non-stressed elements at a constant rate (see also /16/). Let a constant force and zero angle of rotation be given, and let assume that the base of the stamp is flat. The configuration of the body at any instant of time is specified by the coordinate of the tip of the quarter-plane $x = -f(t)$ ($f(t) = 0$, when $\tau_0 \leqslant t \leqslant \tau_1$). We shall assume that the process of accretion does not stop ($\tau_2 = \infty$), i.e. in the limit the body becomes a half-plane, We use formulas (2.12), where we replace r, ρ by x, ξ to change to dimensionless quantities, and we also put $f^*(t^*) = f(t)a^{-1}$ (neglecting the asterisks). The elastic and rheological characteristics of the material are taken from Example 1. We shall also assume that the modes of loading the quarter-plane by the stamp up to the onset of accretion in the first and second example are identical.

Let us determine the contact pressures and the eccentricity of the point of application of the force, ensuring that there is no misalignment of the stamp at different rates of translational accretion. We shall put $P(t) = 1$, $\alpha = \pi/2$, $c = 9$, $\omega(t) = 0$, $g(x) = 0$, $\tau_1 = 2$, $\tau_0 = 1$; 1*) slow translational accretion: $f(t) = t - 2$, $f^{\cdot}(t) = 1$ $(t \geqslant \tau_1)$; 2*) rapid translational accretion: $f(t) = 6(t-2)$, $f^{\cdot}(t) = 6$ $(t \geqslant \tau_1)$ (the rate of accretion $f^{\cdot}(t)$ is 6 times as high as that in the case 1*)).

Since in both cases the quarter-plane loaded by the stamp grows to become a half-plane (in the second case and in the limit as $t \to \infty$), we shall also compare the side and the translational method of accretion from the point of view of their influence of the distribution of contact stresses and the eccentricity. In the graphs shown we shall retain the notation of Example 1, remembering that the methods of accretion are different and, that there is no termination of growth. Apart from that we have complete identity.

Fig.5 shows the limit distribution of the contact stresses at various rates $f^{\cdot}(t)$, the distribution from the instant $\tau_0 = 1$ to the instant $\tau_1 = 2$, and the dependence of the eccentricity on time.

Fig.6 shows the variation with time of the contact stresses q_1 at the left edge of the stamp $(x = 0, 1)$ and q_2 at the right edge $(x = 8, 9)$ for slow and rapid accretion.

We note that in case 1*) the stresses, e.g. at the left edge, increase by a factor of 1.4 and the eccentricity is reduced by a factor of 1.2. In case 2*) these relations are equal to 2.7 and 1.5, respectively. When $t = t^0 \approx 8$ and the rate of accretion is arbitrary, the process becomes a steady-state process and further growth of the body does not in practice affect the contact stresses and the eccentricity. The arguments preceding Note 3 remain valid in this case.

Comparing examples 1 and 2, we find that the method of translational accretion, unlike side accretion, leads to a smoother change in the characteristics, and its influence on the situation under the stamp is weaker. When the forces are constant, a characteristic time t^0 will exist independently of the rate and method of accretion, after which we can neglect the effect of the accretion process on the contact characteristics.

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APPLICATION OF THE PRINCIPLE OF CHOICE TO THE PROBLEM OF THE INITIAL DEVELOPMENT OF SLIP LINES FROM A CORNER POINT*

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Symmetrical problems of initial development near the corner point of the boundary of the body of a plastic zone modelled by two straight slip lines emerging from the vertex are considered under conditions of plane deformation. Functional Wiener-Hopf equations of the problems and their exact analytic solutions are given. The length of the slip lines and the angle of their inclination to the boundary are determined. The principle of choice is used to find the latter. According to this principle, of all possible directions of the development of slip lines, the direction realized corresponds to the maximum value of the rate of dissipation of energy by the body.

The last few years have seen the publication of a number of papers in mechanics of fracture, dealing with problems of initial development within the bodies, near the concentrators, of plastic zones modelled by straight slip lines emerging from the vertex at some angle to the boundary /1-6/. Everyone of these problems reduces to a functional Wiener-Hopf equation, and its solution is used to establish the dependence of the length of the slip line on its angle of inclination to the boundary, the latter being a free parameter. The value of this angle at which the slip line is of maximum length, is taken as the unknown quantity which determines the direction in which the slip line develops.

In the present paper a new, stricter approach is proposed, towards solving the problem of the direction in which the slip lines emerging from the corner point develop. The approach is based on the principle of

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